WEIGHTED INTEGRABILITY OF POLYHARMONIC FUNCTIONS

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Abstract. To address the uniqueness issues associated with the Dirichlet problem for the N-harmonic equation on the unit disk $\mathbb{D}$ in the plane, we investigate the L^p integrability of N-harmonic functions with respect to the standard weights $(1-|z|^2)^\alpha$. The question at hand is the following. If u solves $\Delta^N u = 0$ in $\mathbb{D}$, where Δ stands for the Laplacian, and

$$\int_{\mathbb{D}} |u(z)|^p (1-|z|^2)^\alpha \mathrm{d}A(z) < +\infty,$$

must then $u(z) \equiv 0$? Here, N is a positive integer, α is real, and $0 < p < +\infty$; $\mathrm{d}A$ is the usual area element. The answer will, generally speaking, depend on the triple (N, p, α). The most interesting case is $0 < p < 1$. For a given N, we find an explicit critical curve $p \mapsto \beta(N, p)$ – a piecewise affine function – such that for $\alpha > \beta(N, p)$ there exist non-trivial functions u with $\Delta^N u = 0$ of the given integrability, while for $\alpha \le \beta(N, p)$, only $u(z) \equiv 0$ is possible. We also investigate the obstruction to uniqueness for the Dirichlet problem, that is, we study the structure of the functions in $\mathrm{PH}^p_{N,\alpha}(\mathbb{D})$ when this space is nontrivial. We find a new structural decomposition of the polyharmonic functions – the cellular decomposition – which decomposes the polyharmonic weighted L^p space in a canonical fashion. Corresponding to the cellular expansion is a tiling of part of the (p, α) plane into cells.

In memory of Boris Korenblum

1. Introduction

1.1. **Basic notation.** Let

$$\Delta := \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}, \qquad \mathrm{d}A(z) := \mathrm{d}x\mathrm{d}y,$$

denote the Laplacian and the area element, respectively. Here, $z = x + \mathrm{i}y$ is the standard decomposition into real and imaginary parts. We let $\mathbb{C}$ denote the complex plane, while $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ and $\mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$ denote the unit disk and the unit circle, respectively.

1.2. **Polyharmonic functions.** Given a positive integer N, a function $u : \mathbb{D} \to \mathbb{C}$ is said to be *N-harmonic* (alternative terminology: *polyharmonic of degree $N-1$*) if

$$\Delta^N u = 0$$

holds on $\mathbb{D}$ in the sense of distribution theory. If u is N-harmonic for some N, we say that it is *polyharmonic*. Clearly, 1-harmonic functions are just ordinary harmonic functions. The 2-harmonic functions are of particular interest, and they are usually said to be *biharmonic*. While the Laplacian Δ is associated with a *membrane*, the bilaplacian Δ^2 is associated with a *plate*. There is a sizeable literature related to the bilaplacian, and more generally the N-laplacian Δ^N; see, e.g., the books [8], [10], [12], and the papers [9], [18], [19], [20], [1], [13], [27], [28]. We could

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also mention that the biharmonic Green function was used in [6] to generalize the factorization of L^2 Bergman space functions found in [16] to the L^p setting (see also [17], [3]). Later, it was applied to Hele-Shaw flow on surfaces [24], [23], [22].

1.3. **Uniqueness for the Dirichlet problem.** The Dirichlet problem associated with the N-Laplacian is

$$\begin{cases} \Delta^N u = 0 & \text{on } \mathbb{D}, \\ \partial_{\mathrm{n}}^j u = f_j & \text{on } \mathbb{T} \text{ for } j = 0, 1, \ldots, N-1, \end{cases}$$

where ∂_{n} stands for the (interior) normal derivative. A natural way to interpret this problem is to first construct a function F on $\mathbb{D}$ which encodes the boundary information from the data $f_0, \ldots, f_{N-1}$, and to say that (1.3) asks of u that $\Delta^N u = 0$ and that $u - F$ belongs to a class of functions that decay rapidly to 0 near the boundary $\mathbb{T}$. Often, this decay is understood in terms of Sobolev spaces (see any book on partial differential equations; for a slightly different approach, see, e.g., [5]). A perhaps simpler requirement is to ask that

$$|u(z) - F(z)| = \mathrm{o}((1 - |z|)^{N-1}) \quad \text{as } |z| \to 1^-, \tag{1.1}$$

and it is easy to see that u is uniquely determined by the differential equation $\Delta^N u = 0$ combined with (1.1). If we focus on uniqueness, then by by forming differences we may as well assume $F(z) \equiv 0$. We may then think of (1.1) as

$$(1 - |z|)^{-N+1} |u(z)| \in L_0^\infty(\mathbb{D}), \tag{1.2}$$

where $L_0^\infty(\mathbb{D})$ stands for the closed subspace of $L^\infty(\mathbb{D})$ consisting of functions with limit 0 along $\mathbb{T}$. So (1.2) forces an N-harmonic function to vanish identically. What would happen if we replace $L_0^\infty(\mathbb{D})$ by, for instance $L^p(\mathbb{D})$, for some p, $0 < p < +\infty$? We would expect that we ought to change the exponent in the distance to the boundary, but by how much? More precisely, we would like to know for which real (negative) α the implication

$$(1 - |z|)^\alpha |u(z)| \in L^p(\mathbb{D}) \implies u \equiv 0 \tag{1.3}$$

holds for all N-harmonic functions u. Here, we shall obtain the complete answer to this question. We obtain an explicit expression $\beta(N, p)$ such that the implication (1.3) holds if and only if $\alpha \le \beta(N, p)/p$. When $\alpha > \beta(N, p)/p$, when we have non-uniqueness, we study the source of obstruction to uniqueness. We obtain an understanding of those obstructions in terms of a decomposition of N-harmonic functions – which we call the *cellular decomposition* – associated with a nontrivial factorization of the N-Laplacian.

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2. The Almansi expansion and weighted Lebesgue spaces in the disk

2.1. **Some additional notation.** For $z_0 \in \mathbb{C}$ and positive real r, let $\mathbb{D}(z_0, r)$ denote the open disk centered at z_0 with radius r; moreover, we let $\mathbb{T}(z_0, r)$ denote the boundary of $\mathbb{D}(z_0, r)$ (which is a circle). We let $\mathrm{d}(z) = |\mathrm{d}z|$ denote arc length measure on curves (usually on circles such as $\mathbb{T} = \mathbb{T}(0, 1)$).

The complex differentiation operators

$$\partial_z := \frac{1}{2}\Big(\frac{\partial}{\partial x} - \mathrm{i}\frac{\partial}{\partial y}\Big), \qquad \bar\partial_z := \frac{1}{2}\Big(\frac{\partial}{\partial x} + \mathrm{i}\frac{\partial}{\partial y}\Big),$$

will be useful. Here, $z = x + \mathrm{i}y$ is the standard decomposition of a complex number into real and imaginary parts. It is easy to check that $\Delta = 4\partial_z\bar\partial_z$. If we let

$$\nabla = \nabla_z := \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)$$

denote the gradient, then we find that

$$|\nabla_z u|^2 = 2\Big(|\partial_z u|^2 + |\bar\partial_z u|^2\Big). \tag{2.1}$$

2.2. **The Almansi expansion and the extension of a polyharmonic function.** The classical *Almansi expansion* (or *Almansi representation*) asserts that u is N-harmonic if and only if it is of the form

$$u(z) = u_0(z) + |z|^2 u_1(z) + \cdots + |z|^{2N-2} u_{N-1}(z), \tag{2.2}$$

where all the functions u_j are harmonic in $\mathbb{D}$ (see, e.g., Section 32 of [8]). The harmonic functions u_j which appear in the Almansi expansion (2.2) are uniquely determined by the given N-harmonic function u, as is easy to see from the Taylor expansion of u at the origin, by appropriate grouping of the terms. This allows us to define uniquely the *extension operator* $\mathbf{E}$:

$$\mathbf{E}[u](z, \varrho) = u_0(z) + \varrho^2 u_1(z) + \cdots + \varrho^{2N-2} u_{N-1}(z); \tag{2.3}$$

the function $\mathbb{E}[u]$ will be referred to as *the extension* of u. The extension $\mathbb{E}[u](z,\varrho)$ has the following properties:

(a) it is an even polynomial of degree $2N-2$ in the variable ϱ,

(b) it is harmonic in the variable z, and

$$\mathbf{E}[u](z, |z|) \equiv u(z). \tag{2.4}$$

In fact, the above properties (a)–(b) and (2.4) characterize the extension operator $\mathbf{E}$.

As for notation, we write $\mathrm{PH}_N(\mathbb{D})$ for the linear space of all N-harmonic function in the unit disk $\mathbb{D}$.

2.3. **The standard weighted Lebesgue spaces.** Let $u : \mathbb{D} \to \mathbb{C}$ be a (Borel) measurable function. Given reals p, α with $0 < p < +\infty$, we consider the Lebesgue space $L^p_\alpha(\mathbb{D})$ of (equivalence classes of) functions u with

$$\|u\|^p_{p,\alpha} := \int_{\mathbb{D}} |u(z)|^p (1 - |z|^2)^\alpha \,\mathrm{d}A(z) < +\infty. \tag{2.5}$$

These spaces are standard in the Bergman space context [21]. For $1 \le p < +\infty$, they are Banach spaces, and for $0 < p < 1$, they are quasi-Banach spaces. Clearly, we have the inclusion

$$L^p_\alpha(\mathbb{D}) \subset L^p_{\alpha'}(\mathbb{D}) \quad \text{for} \ \ \alpha < \alpha'. \tag{2.6}$$

2.4. **The L^p-type of a measurable function.** We use the standard weighted Lebesgue spaces to define the concept of the L^p-type of a function.

Definition 2.1. ($0 < p < +\infty$) For a Borel measurable function $u : \mathbb{D} \to \mathbb{C}$, let its *$L^p$-type* be the number

$$\beta_p(u) := \inf\{\alpha \in \mathbb{R} : \ u \in L^p_\alpha(\mathbb{D})\},$$

if the infimum is taken over a non-empty collection. If instead $u \notin L^p_\alpha(\mathbb{D})$ for every $\alpha \in \mathbb{R}$, we write $\beta_p(u) := +\infty$.

The L^p-type measures the boundary growth or decay of the given function u. In particular, it is rather immediate that if u has compact support in $\mathbb{D}$, then its L^p-type equals $\beta_p(u) = -\infty$ for all p, $0 < p < +\infty$. It is a consequence of Hölder's inequality that for a fixed u, the function $p \mapsto \beta_p(u)$ is convex (interpreted liberally).

Here, we want to study the p-type in the context of the spaces of N-harmonic functions $\mathrm{PH}_N(\mathbb{D})$. If we think of the elements of the space $\mathrm{PH}_N(\mathbb{D})$ as physical states, we may interpret the L^p-type $\beta_p(f)$ as the "p-temperature" of the state $f \in \mathrm{PH}_N(\mathbb{D})$. If we fix N and freeze the system – i.e., we consider only states of low "p-temperature" – we should expect that the degrees of freedom are reduced, and eventually, only the trivial state 0 would remain. The "p-temperature" at which this transition occurs is the critical "p-temperature" for the given N.

In mathematical terms, we shall be concerned with the following problem.

Problem 2.2. Given a positive integer n and a p with $0 < p < +\infty$, what is the value of

$$\beta(N,p) := \inf\left\{\beta_p(f) : f \in \mathrm{PH}_N(\mathbb{D}) \setminus \{0\}\right\} ? \tag{2.7}$$

In other words, what is the smallest possible L^p-type of a non-trivial function $f \in \mathrm{PH}_N(\mathbb{D})$? Moreover, is the above infimum attained (i.e., is it a minimum)?

We call the function $p \mapsto \beta(N,p)$ the *critical integrability type curve for the N-harmonic functions*, and the function $(N,p) \mapsto \beta(N,p)$ the *critical integrability type curves for the polyharmonic functions*.

The notion of the critical integrability type is rather parallel to Makarov's integral means spectrum in the context of bounded univalent functions [26] (see also [25]); there, however, a "sup" is used instead of an "inf" in the formula analogous to (2.7), so Makarov's integral means spectrum is automatically convex, which is not true about the critical integrability type curve (see remark below).

Remark 2.3. (a) Since we take an infimum over a collection of f, the property that $p \mapsto \beta_p(f)$ is convex does not carry over to $p \mapsto \beta(N,p)$; indeed, examples will show that $p \mapsto \beta(N,p)$ fails to be convex.

(b) If we were to replace the "inf" with a "sup" in the above definition (2.7), we would not obtain an interesting concept, as it is easy to construct a harmonic function f on $\mathbb{D}$ with $\beta_p(f) = +\infty$ for all p, $0 < p < +\infty$.

2.5. **The Almansi expansion and the boundary decay of polyharmonic functions.** The Almansi expansion (2.2) can be expressed in the following form:

$$u(z) = v_0(z) + (1-|z|^2)v_1(z) + \cdots + (1-|z|^2)^{N-1}v_{N-1}(z), \tag{2.8}$$

where all the functions v_j are harmonic in $\mathbb{D}$. In terms of the functions u_j of (2.2), the functions v_j are given as

$$v_j = (-1)^j \sum_{k=j}^{N-1} \binom{k}{j} u_k.$$

In view of (2.8), we might be inclined to believe that the functions u with

$$v_0 = v_1 = \cdots = v_{N-2} = 0,$$

that is,

$$u = (1-|z|^2)^{N-1} v_{N-1}(z), \tag{2.9}$$

should be the smallest near the boundary. To our surprise, we find that this is not true, at least if we understand the question in terms of Problem 2.2 with $0 < p < \frac{1}{3}$. Somehow the functions

$v_0, \ldots, v_{N-1}$ can cooperate to produce non-trivial functions which decay faster than functions of the type (2.9). We think of this phenomenon as an *entanglement* (see Section 3 for more details).

2.6. **The standard weighted Lebesgue spaces of polyharmonic functions.** We put

$$\mathrm{PH}^p_{N,\alpha}(\mathbb{D}) := \mathrm{PH}_N(\mathbb{D}) \cap L^p_\alpha(\mathbb{D}),$$

and endow it with the norm or quasi-norm structure of $L^p_\alpha(\mathbb{D})$. This is the subspace of $L^p_\alpha(\mathbb{D})$ consisting of N-harmonic functions. It turns out that it is a closed subspace; this is rather non-trivial for $0 < p < 1$, even for $N = 1$. Actually, the proof is based on a property which we will refer to as *Hardy-Littlewood ellipticity of the Laplacian* (cf. [14]; see also [11], pp. 121–123).

2.7. **The harmonic case ($N = 1$).** In [2], Aleksandrov studied essentially our Problem 2.2 in the case of $N = 1$ (harmonic functions). To explain the result, we make some elementary calculations. The constant function $U_0(z) \equiv 1$ is harmonic, and

$$U_0 = 1 \in \mathrm{PH}^p_{1,\alpha}(\mathbb{D}) \iff \alpha > -1. \tag{2.10}$$

This shows that

$$\beta_p(U_0) = -1. \tag{2.11}$$

Next, we turn to the Poisson kernel at 1,

$$U_1(z) = P(z, 1) = \frac{1 - |z|^2}{|1 - z|^2}.$$

We shall need the following lemma. The formulation involves the standard Pochhammer symbol notation $(x)_j := x(x + 1) \cdots (x + j - 1)$.

Lemma 2.4. *Let a, b be two real parameters with $b \geq 0$. If we put*

$$I(a, b) := \int_{\mathbb{D}} \frac{(1 - |z|^2)^a}{|1 - z|^{2b}} \mathrm{d}A(z),$$

then $I(a, b) = +\infty$ if $a \leq -1$ or if both $b > 0$ and $a \leq 2(b - 1)$. If, on the other hand, $a > -1$ and $b = 0$, then $I(a, b) = \pi/(a + 1)$. Moreover, if $a > -1$, $b > 0$, and $a > 2(b - 1)$, then $I(a, b) < +\infty$, with value

$$I(a, b) = \pi \sum_{j=0}^{+\infty} \frac{[(b)_j]^2}{j!(a + 1)_{j+1}}.$$

Proof. By Taylor expansion and polar coordinates, we find that

$$\begin{aligned} I(a, b) = \int_{\mathbb{D}} |1 - z|^{-2b}(1 - |z|^2)^a \mathrm{d}A(z) &= \int_{\mathbb{D}} \left| \sum_{j=0}^{+\infty} \frac{(b)_j}{j!} z^j \right|^2 (1 - |z|^2)^a \mathrm{d}A(z) \\ &= \sum_{j=0}^{+\infty} \frac{[(b)_j]^2}{[j!]^2} \int_{\mathbb{D}} |z|^{2j}(1 - |z|^2)^a \mathrm{d}A(z) = \pi \sum_{j=0}^{+\infty} \frac{[(b)_j]^2}{[j!]^2} \int_0^1 t^j (1 - t)^a \mathrm{d}t. \end{aligned}$$

The (Beta) integral on the right-hand side diverges for $a \leq -1$, so that $I(a, b) = +\infty$ then. For $a > -1$, the Beta integral is quickly evaluated, and we obtain that

$$I(a, b) = \pi \sum_{j=0}^{+\infty} \frac{[(b)_j]^2}{j!(a + 1)_{j+1}},$$

and for $b > 0$ the sum on the right-hand side converges if and only if $a > 2(b - 1)$, by the standard approximate formulae for the Gamma function. $\square$

We see from Lemma 2.4 that

$$U_1 = P(\cdot, 1) \in \mathrm{PH}^p_{1,\alpha}(\mathbb{D}) \iff \alpha > \max\{p-2, -1-p\}, \tag{2.12}$$

which implies that its L^p-type is

$$\beta_p(U_1) = \max\{p-2, -1-p\}. \tag{2.13}$$

Now, in view of (2.11) and (2.13), the critical integrability type $\beta(1,p)$ satisfies

$$\beta(1,p) \le \min\{\beta_p(U_0), \beta_p(U_1)\} = \min\Big\{-1, \max\{p-2, -1-p\}\Big\}.$$

The profound work of Aleksandrov [2] is mainly concerned with harmonic functions in the unit ball of $\mathbb{R}^d$ and the maximal possible rate of decay of the L^p-integral on concentric spheres $x_1^2 + \cdots + x_d^2 = r^2$. In the rather elementary planar case $d = 2$, it gives to the following result (see also Suzuki [32]).

Theorem 2.5. *We have that* $\beta(1,p) = \min\{-1, \max\{p-2, -1-p\}\}$ *for all* p, $0 < p < +\infty$.

This has the interpretation that the constant function $U_0 = 1$ and the Poisson kernel $U_1 = P(\cdot, 1)$ are jointly extremal for the problem of determining the critical L^p-type for harmonic functions. Indeed, for $0 < p \le 1$, we have $\beta(1,p) = \beta_p(U_1)$, while for $1 \le p < +\infty$, we have instead $\beta(1,p) = \beta_p(U_0)$. The function $\beta(1,p)$ is therefore continuous and piecewise affine: $\beta(1,p) = -1-p$ for $0 < p \le \frac{1}{2}$, $\beta(1,p) = p-2$ for $\frac{1}{2} \le p \le 1$, and $\beta(1,p) = -1$ for $1 \le p < +\infty$.

3. Main results

3.1. **Characterization of the critical integrability type curve.** We let $b_j(p)$ be the function

$$b_{j,N}(p) := \max\Big\{-1-(j+N-1)p, -2+(j-N+1)p\Big\}, \qquad j = 1, \ldots, N, \tag{3.1}$$

whose graph is piecewise affine, while for $j = 0$ we put

$$b_{0,N}(p) := -1-(N-1)p, \tag{3.2}$$

which is affine. It is easy to check that

$$b_{j,N+1}(p) + p = b_{j,N}(p), \qquad j = 0, \ldots, N. \tag{3.3}$$

We present our first main theorem.

Theorem 3.1. $(0 < p < +\infty)$ *For* $N = 1, 2, 3, \ldots$ *and for real* α, *we have that*

$$\mathrm{PH}^p_{N,\alpha}(\mathbb{D}) = \{0\} \iff \alpha \le \min_{j: 0 \le j \le N} b_{j,N}(p).$$

As an immediate consequence, we obtain the explicit evaluation of the critical integrability type curve.

Corollary 3.2. *The critical integrability type for the polyharmonic functions is given by*

$$\beta(N,p) = \min_{j: 0 \le j \le N} b_{j,N}(p),$$

for $0 < p < +\infty$ *and* $N = 1, 2, 3, \ldots$.

Being the minimum of a finite number of continuous piecewise affine functions, the function $p \mapsto \beta(N,p)$ is then continuous and piecewise affine. It is easy to check that

$$\beta(N,p) = \min_{j:0\le j\le N} b_{j,N}(p) = \min\{b_{0,N}(p), b_{1,N}(p)\} \quad \text{for } \tfrac{1}{3} \le p < +\infty, \tag{3.4}$$

and since $b_{0,N}(p)$ and $b_{1,N}(p)$ equal the L^p-type of the N-harmonic functions $z \mapsto (1-|z|^2)^{N-1}$ and $z \mapsto (1-|z|^2)^N/|1-z|^2$, we may interpret this as concrete support for the intuition of Subsection 2.5 (based on the Almansi expansion) for $\frac{1}{3} \le p < +\infty$. However, it is again easy to verify that

$$\beta(N,p) = \min_{j:0\le j\le N} b_{j,N}(p) < \min\{b_{0,N}(p), b_{1,N}(p)\} \quad \text{for } 0 < p < \tfrac{1}{3},$$

so the intuition fails then, and we interpret this as the appearance of entanglement.

Remark 3.3. We draw the graphs of $p \mapsto \beta(N,p)$, for $N = 2, 3$, in Figures 3.1 ($N = 2$) and 3.2 ($N = 3$), respectively.

We prove Theorem 3.1 in Section 5; the work is based on the property of the N-Laplacian which we call Hardy-Littlewood ellipticity (see Section 4).

3.2. **The structure of polyharmonic functions.** We pass to the study of the structure of the space $\mathrm{PH}^p_{N,\alpha}(\mathbb{D})$ when the space contains nontrivial elements. We fix an integer $N = 2, 3, 4, \ldots$, and let $\mathcal{A}_N \subset \mathbb{R}^2$ be the open set

$$\mathcal{A}_N := \Big\{(p,\alpha) : \ 0 < p < +\infty \text{ and } \min_{j:0\le j\le N} b_{j,N}(p) < \alpha\Big\};$$

then the assertion of Theorem 3.1 is equivalent to the statement

$$(p,\alpha) \in \mathcal{A}_N \iff \mathrm{PH}^p_{N,\alpha}(\mathbb{D}) \ne \{0\}.$$

We will at times refer to $\mathcal{A}_N$ as the *admissible region*. Let us introduce the second order elliptic partial differential operator $\mathbf{L}_\theta$ indexed by a real parameter θ,

$$\mathbf{L}_\theta[u](z) := (1-|z|^2)\Delta u(z) + 4\theta[z\partial_z u(z) + \bar z\bar\partial_z u(z)] - 4\theta^2 u(z). \tag{3.5}$$

We note that (when desirable) the complex derivatives can be eliminated by considering polar coordinates:

$$r\partial_r = z\partial_z + \bar z\bar\partial_z.$$

To simplify the notation, we let $\mathbf{M}$ denote the multiplication operator given by

$$\mathbf{M}[v](z) := (1-|z|^2)v(z).$$

We observe that for $j = 0, \ldots, N-1$,

$$\mathbf{M}^j[v] \in L^p_\alpha(\mathbb{D}) \iff v \in L^p_{\alpha+jp}(\mathbb{D}), \tag{3.6}$$

so that

$$\mathbf{M}^j[v] \in \mathrm{PH}^p_{N,\alpha}(\mathbb{D}) \text{ and } \Delta^{N-j}v = 0 \iff v \in \mathrm{PH}^p_{N-j,\alpha+jp}(\mathbb{D}). \tag{3.7}$$

The following is our main structure theorem for the N-harmonic functions.

Theorem 3.4. (The cellular decomposition theorem) *Let α be real, and let p be positive. Then, for $N = 1, 2, 3, \ldots$, every $u \in \mathrm{PH}^p_{N,\alpha}(\mathbb{D})$ has a unique decomposition*

$$u = w_0 + \mathbf{M}[w_1] + \cdots + \mathbf{M}^{N-1}[w_{N-1}],$$

where each term $\mathbf{M}^j[w_j]$ is in $\mathrm{PH}^p_{N,\alpha}(\mathbb{D})$, while the functions w_j are $(N-j)$-harmonic and solve the partial differential equation $\mathbf{L}_{N-j-1}[w_j] = 0$ on $\mathbb{D}$, for $j = 0, \ldots, N-1$.

Remark 3.5. In the context of Theorem 3.4, the mapping $u \mapsto \mathbf{M}^j[w_j]$ defines an idempotent operator $\mathbf{P}_j$ on $\mathrm{PH}^p_{N,\alpha}(\mathbb{D})$, for $j = 0, \ldots, N-1$. It is clear from the proof of the theorem that each $\mathbf{P}_j$ acts continuously on $\mathrm{PH}^p_{N,\alpha}(\mathbb{D})$.

Though reminiscent of the alternative Almansi expansion (2.8) (compare with Pavlović [31]), the expansion of Theorem 3.4 is different, because here, the functions w_j are not assumed harmonic, instead they solve the partial differential equation $\mathbf{L}_{N-j-1}[w_j] = 0$. We suggest to call the expansion of Theorem 3.4 the *cellular decomposition*, because it relates to the cell structure of the admissible region $\mathcal{A}_N$ (see below). It is a crucial feature of Theorem 3.4 that each term of the decomposition remains in the space $\mathrm{PH}^p_{N,\alpha}(\mathbb{D})$. This means that we may analyze each term separately.

3.3. **The structure of polyharmonic functions: the entangled and unentangled regions.** We consider the following relatively closed bounded subset of the admissible region $\mathcal{A}_N$

$$\mathcal{E}_N := \Big\{(p,\alpha) \in \mathcal{A}_N : \ 0 < p < \tfrac{1}{3} \text{ and } \alpha \le -1 - Np\Big\}.$$

We will refer to $\mathcal{E}_N$ as the *entangled region*. The complement $\mathcal{N}_N := \mathcal{A}_N \setminus \mathcal{E}_N$ is then open, and we call it the *unentangled region*. These two regions will be further subdivided into smaller units which we refer to as *cells*. In particular, the unentangled region has a *principal unentangled cell*,

$$\mathcal{N}_N^{(1)} := \Big\{(p,\alpha) \in \mathcal{N}_N : \ \tfrac{1}{3} < p \text{ and } \alpha \le \min\{-2 + (3-N)p, -1 + (2-N)p\}\Big\},$$

which is relatively closed in $\mathcal{N}_N$.

Proposition 3.6. *We have that* $(p,\alpha) \in \mathcal{A}_N$ *belongs to the entangled region* $\mathcal{E}_N$ *if and only if*

$$u = \mathbf{M}^{N-1}[v] \in \mathrm{PH}^p_{N,\alpha}(\mathbb{D}) \ \textit{for harmonic} \ v \quad \Longrightarrow \quad u = 0.$$

In other words, the entangled region describes where the space $\mathrm{PH}^p_{N,\alpha}(\mathbb{D})$ contains no non-trivial functions of the form $(1-|z|^2)^{N-1}v(z)$ with v harmonic. The principal unentangled cell has a similar-looking characterization.

Proposition 3.7. *We have that* $(p,\alpha) \in \mathcal{N}_N$ *belongs to the principal unentangled cell* $\mathcal{N}_N^{(1)}$ *if and only if*

$$u \in \mathrm{PH}^p_{N,\alpha}(\mathbb{D}) \quad \Longrightarrow \quad u = \mathbf{M}^{N-1}[v] \ \textit{for some harmonic} \ v.$$

Recent work of Olofsson [29] shows the following. We assume that θ is a nonnegative integer for technical reasons; the result may well be true for general real $\theta > -\frac{1}{2}$.

Proposition 3.8. $(0 < p < +\infty)$ *Fix a nonnegative integer* θ *and a real parameter* α*. For a function* $v \in L^p_\alpha(\mathbb{D})$*, the following are equivalent:*
(i) *The function* v *is continuous in* $\mathbb{D}$ *and solves* $\mathbf{L}_\theta[v] = 0$ *in* $\mathbb{D}$ *in the sense of distributions, and*
(ii) *The function* v *is of the form*

$$v(z) = \frac{1}{2\pi}\int_{\mathbb{T}} \frac{(1-|z|^2)^{2\theta+1}}{|1-z\bar{\xi}|^{2\theta+2}} f(\xi)\mathrm{d}s(\xi), \qquad z \in \mathbb{D},$$

for some distribution f *on* $\mathbb{T}$*, where the integral is understood in the sense of distribution theory.*

Proposition 3.8 tells us that the functions w_j of Theorem 3.4 may expressed as Poisson-type integrals of their distributional boundary values (compare with the remark below).

Remark 3.9. In the context of Proposition 3.8, the boundary distribution f equals $c(\theta)$ times the limit of of the dilates $v_r(\zeta) = v(r\zeta)$, where $\zeta \in \mathbb{T}$, as $r \to 1^-$. Here, $c(\theta)$ is the constant

$$c(\theta) := \frac{\Gamma(1+\theta)^2}{\Gamma(1+2\theta)}.$$

3.4. **The cells of the admissible region: entangled and unentangled.** To properly analyze all the cells of the admissible region, we should first introduce a modification of the functions $b_{j,N}(p)$ given by (3.1). So, we write

$$\text{(3.8)} \quad a_{j,N}(p) := \min\left\{b_{j,N}(p), -1+(j-N)p\right\}$$
$$= \begin{cases} -1-(j+N-1)p, & 0<p<1/(2j), \\ -2+(j-N+1)p, & 1/(2j)\le p<1, \\ -1+(j-N)p & p\ge 1 \end{cases} \qquad j=1,\ldots,N,$$

and observe that $a_{j,N}(p) = b_{j,N}(p)$ for $0 < p \le 1$, while $a_{j,N}(p) = -1 + (j-N)p$ for $p \ge 1$. We check that the graph of $p \mapsto a_{j,N}(p)$ is continuous and piecewise affine. The analogue of (3.3) reads as follows:

$$\text{(3.9)} \qquad a_{j,N+1}(p) + p = a_{j,N}(p), \qquad j = 1,\ldots,N.$$

If we draw all the curves $\alpha = a_{j,N}(p)$ within the admissible region $\mathcal{A}_N$, they slice up the region into pieces we call *cells*. To make this precise, we proceed as follows. For a point $(p,\alpha) \in \mathcal{A}_N$, we put

$$J(p,\alpha) := \left\{j \in \{0,\cdots,N-1\} : \ \alpha > a_{N-j,N}(p)\right\},$$

which defines a function from $\mathcal{A}_N$ to the collection of all subsets of $\{0,\cdots,N-1\}$. The *admissible cells* are the level sets (i.e., the sets of constancy) for this function J. As mentioned above, the admissible cells contained in the entangled region are called *entangled cells*, while those contained in the unentangled region are called *unentangled cells*; all admissible cells belong to one of these categories. We draw the cell decomposition for $N = 2,3$ in Figures 3.1 and 3.2.

We can now improve upon Theorem 3.4.

Theorem 3.10. *Suppose $(p,\alpha) \in \mathcal{A}_N$. Then every $u \in \mathrm{PH}^p_{N,\alpha}(\mathbb{D})$ has a unique decomposition*

$$u = \sum_{j\in J(p,\alpha)} \mathbf{M}^j[w_j],$$

where each term $\mathbf{M}^j[w_j]$ is in $\mathrm{PH}^p_{N,\alpha}(\mathbb{D})$, while the functions w_j are $(N-j)$-harmonic and solve the partial differential equation $\mathbf{L}_{N-j-1}[w_j] = 0$ on $\mathbb{D}$, for $j \in J(p,\alpha)$.

Remark 3.11. (i) The main improvement in Theorem 3.10 is that we may now specify which terms of the cellular decomposition must necessarily vanish. The theorem is sharp, in the sense that each term $\mathbf{M}^j[w_j]$ with $j \in J(p,\alpha)$ is allowed to be nontrivial. This is easy to see easily by considering Dirac point masses f or uniform density (constant) $f = 1$ in Proposition 3.8 (compare with Lemma 5.2 below). For details, see Subsection 7.1.
(ii) The cellular decomposition can be likened to the decomposition of a vector with respect to a given basis. Theorem 3.10 tells us which pieces of the "basis" are being used in a given cell. The number of elements of the set $J(p,\alpha) \subset \{0,\ldots,N-1\}$ is like the "dimension" of the subspace spanned by the given vectors. Perhaps "degrees of freedom" would be a better term.

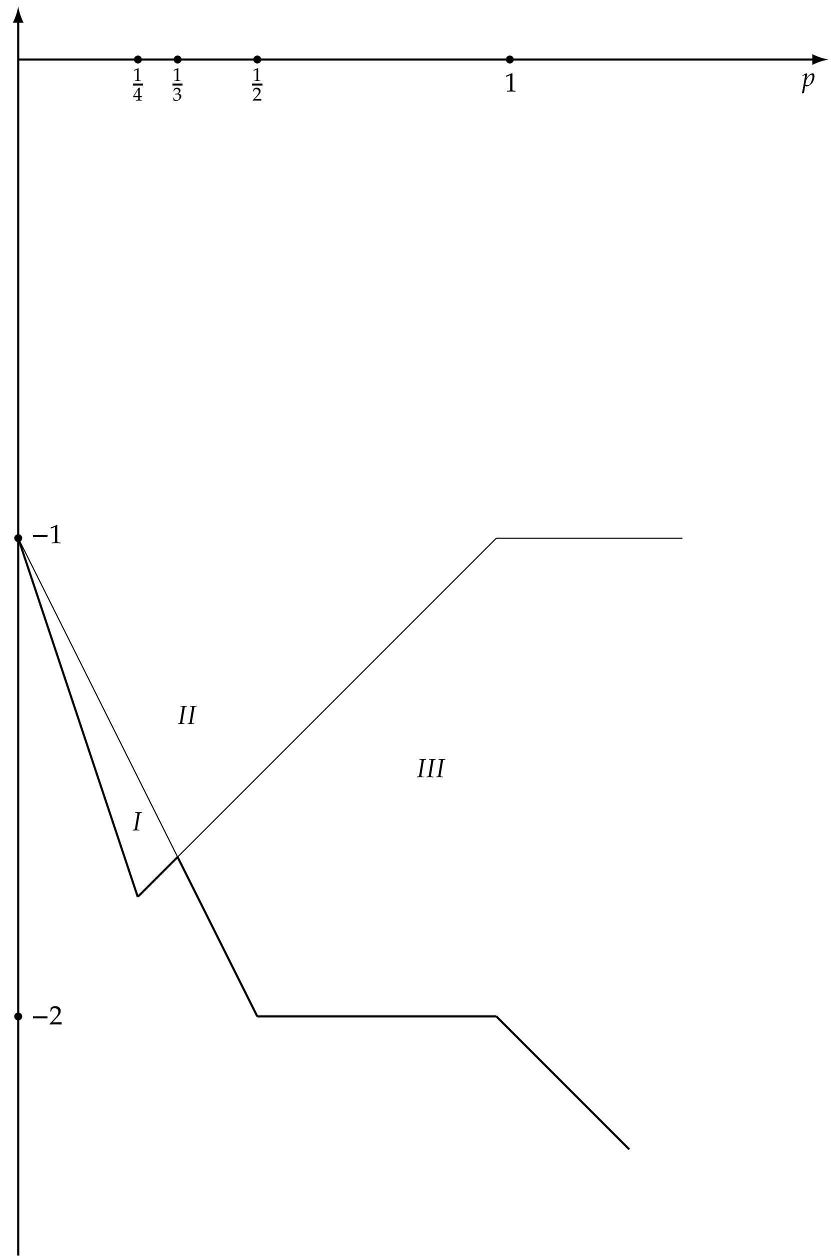

FIGURE 3.1. The graph of $\beta(2,p)$ (in thick style), plus indication of admissible cells. Here, I is an entangled cell, while II and III are unentangled cells (III is principal).

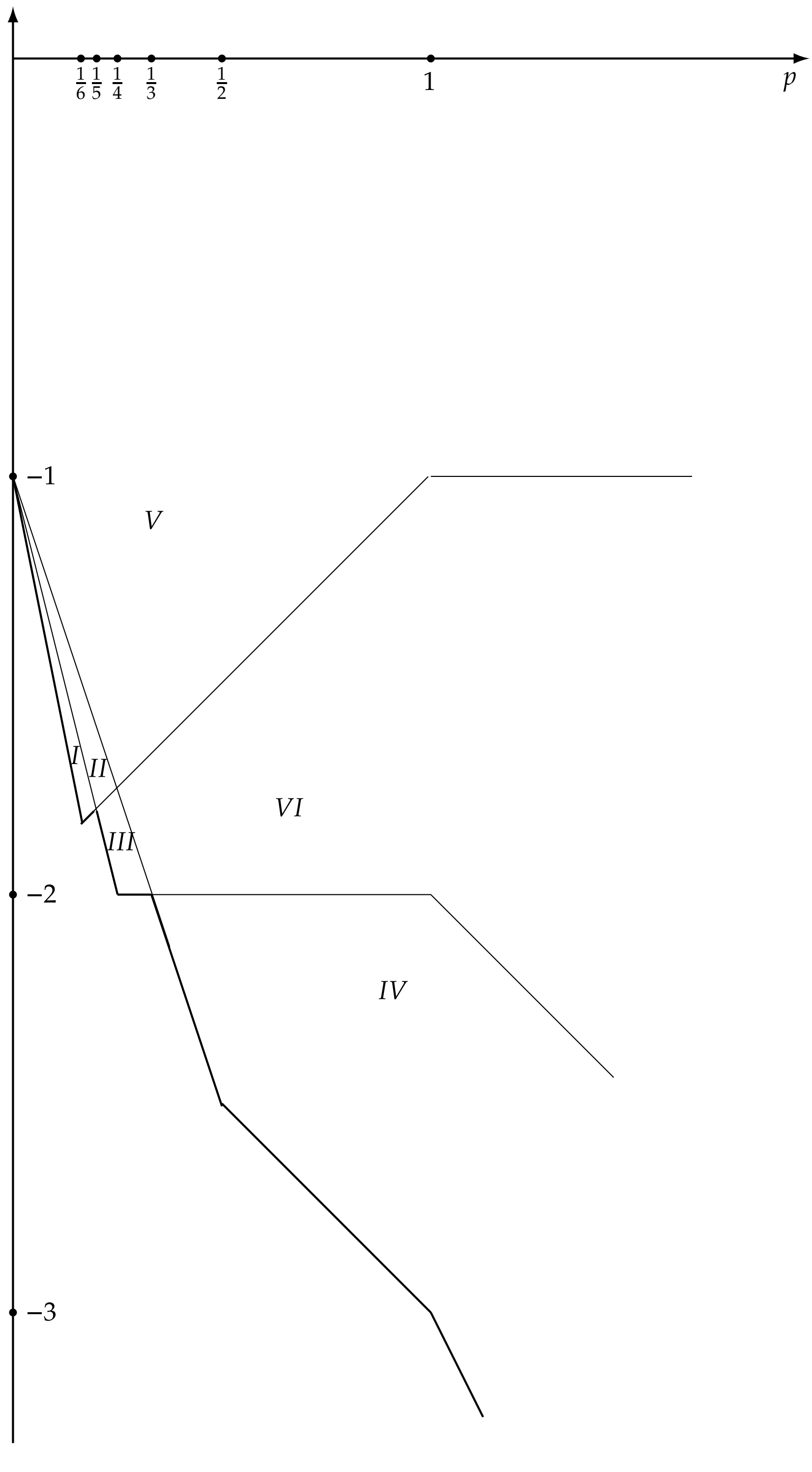

FIGURE 3.2. The graph of $\beta(3,p)$ (in thick style), plus indication of admissible cells. Here, I, II, and II are entangled cells, while IV, V, and VI are unentangled cells (IV is principal).

4. The polyharmonic Hardy-Littlewood ellipticity and its applications

4.1. **Representation of the extension in terms of the Poisson kernel.** If a function $f:\mathbb{D}(0,r)\to\mathbb{C}$ is harmonic and extends continuously to the boundary, then Poisson's formula supplies the representation

$$f(z)=\frac{r^2-|z|^2}{2\pi r}\int_{\mathbb{T}(0,r)}|z-\zeta|^{-2}f(\zeta)\mathrm{d}s(\zeta),\qquad z\in\mathbb{D}(0,r),\tag{4.1}$$

where we recall that ds stands for arc length measure.

We suppose that $u\in\mathrm{PH}_N(\mathbb{D})$, i.e. that u is N-harmonic in $\mathbb{D}$, and recall the notation $\mathbf{E}[u](z,\varrho)$ for the extension of u, given by (2.3). With $f(z)=\mathbf{E}[u](z,\varrho)$ and $r=\varrho$, we obtain from (4.1) that

$$\mathbf{E}[u](z,\varrho)=\frac{\varrho^2-|z|^2}{2\pi\varrho}\int_{\mathbb{T}(0,\varrho)}|z-\zeta|^{-2}u(\zeta)\mathrm{d}s(\zeta),\qquad z\in\mathbb{D}(0,\varrho),\tag{4.2}$$

for each ϱ with $0<\varrho<1$, since $\mathbf{E}[u](\zeta,\varrho)=u(\zeta)$ for $\zeta\in\mathbb{T}(0,\varrho)$ by (2.4). In particular, we obtain in an elementary fashion that

$$|\mathbf{E}[u](z,\varrho)|\le\frac{1}{2\pi\varrho}\,\frac{\varrho+|z|}{\varrho-|z|}\int_{\mathbb{T}(0,\varrho)}|u(\zeta)|\mathrm{d}s(\zeta),\qquad z\in\mathbb{D}(0,\varrho),\tag{4.3}$$

4.2. **Lagrangian interpolation of the extension of a polyharmonic function.** We suppose that $u\in\mathrm{PH}_N(\mathbb{D})$, i.e. that u is N-harmonic in $\mathbb{D}$, and recall the notation $\mathbf{E}[u](z,\varrho)$ for the extension of u, given by (2.3). The polynomial nature of $\varrho\mapsto\mathbf{E}[u](z,\varrho)$ makes it amenable to Lagrangian interpolation. Indeed, if we supply real values ϱ_j, where $j=1,\dots,N$, with $0<\varrho_1<\varrho_2<\cdots<\varrho_N<1$, then

$$\mathbf{E}[u](z,\varrho)=\sum_{j=1}^{N}\mathbf{E}[u](z,\varrho_j)\,L_j(\varrho),\tag{4.4}$$

where $L_j(\varrho)$ is the even interpolation polynomial

$$L_j(\varrho):=\prod_{k:k\neq j}\frac{\varrho^2-\varrho_k^2}{\varrho_j^2-\varrho_k^2};\tag{4.5}$$

here, it is tacitly assumed that the product runs over the set of integers $k\in\{1,\dots,N\}$ (with the exception of j). In particular, with $\varrho=|z|$, (2.4) and (4.4) together give the representation

$$u(z)=\sum_{j=1}^{N}\mathbf{E}[u](z,\varrho_j)\,L_j(|z|),\qquad z\in\mathbb{D}.\tag{4.6}$$

If we use both (4.2) and (4.6), we see that

$$u(z)=\sum_{j=1}^{N}L_j(|z|)\,\frac{\varrho_j^2-|z|^2}{2\pi\varrho_j}\int_{\mathbb{T}(0,\varrho_j)}|z-\zeta|^{-2}u(\zeta)\mathrm{d}s(\zeta),\qquad z\in\mathbb{D}(0,\varrho_1).\tag{4.7}$$

We can think about (4.7) as a polyharmonic analogue of the Poisson representation. Let us introduce the even polynomial

$$M_j(\varrho):=L_j(\varrho)\,\frac{\varrho_j^2-\varrho^2}{2\varrho_j}=-\frac{\prod_k(\varrho^2-\varrho_k^2)}{2\varrho_j\prod_{k:k\neq j}(\varrho_j^2-\varrho_k^2)},\tag{4.8}$$

which is of degree $2n$ and solves the interpolation problem

$$M_j(\varrho_k)=0\quad\text{for all }\;k,\qquad M_j'(\varrho_j)=1.$$

In terms of the functions M_j, the formula (4.6) simplifies:

$$u(z) = \frac{1}{\pi}\sum_{j=1}^{N} M_j(|z|)\int_{\mathbb{T}(0,\varrho_j)} |z-\zeta|^{-2}u(\zeta)\mathrm{d}s(\zeta), \qquad z \in \mathbb{D}(0,\varrho_1). \tag{4.9}$$

We may of course apply the gradient to both sides:

$$\nabla u(z) = \frac{1}{\pi}\sum_{j=1}^{N}\left\{[\nabla M_j(|z|)]\int_{\mathbb{T}(0,\varrho_j)} |z-\zeta|^{-2}u(\zeta)\mathrm{d}s(\zeta) + M_j(|z|)\int_{\mathbb{T}(0,\varrho_j)}[\nabla_z|z-\zeta|^{-2}]u(\zeta)\mathrm{d}s(\zeta)\right\}, \tag{4.10}$$

for $z \in \mathbb{D}(0,\varrho_1)$. We write $\varrho_1 = \vartheta$, and let δ denote the quantity

$$\delta := \min_{j,k:j\neq k}|\varrho_j^2 - \varrho_k^2|. \tag{4.11}$$

An elementary calculation then gives that

$$|M_j(|z|)| \le \delta^{-N+1}(\vartheta - |z|) \ \text{ and } \ |\nabla M_j(|z|)| \le N\,\delta^{-N+1} \quad \text{for} \quad z \in \mathbb{D}(0,\vartheta), \tag{4.12}$$

while another gives that

$$|z-\zeta|^{-2} \le (\vartheta-|z|)^{-2} \ \text{ and } \ |\nabla_z|z-\zeta|^{-2}| \le 2(\vartheta-|z|)^{-3} \quad \text{for} \quad z\in\mathbb{D}(0,\vartheta),\ \ \zeta \in \cup_j\mathbb{T}(0,\varrho_j). \tag{4.13}$$

As we implement these estimates into (4.9) and (4.10), we obtain the following estimates.

Lemma 4.1. *If* $0 < \vartheta = \varrho_1 < \cdots < \varrho_N < 1$, *and* δ *is given by* (4.11), *then*

$$|u(z)| \le \frac{\delta^{-N+1}}{(\vartheta-|z|)\pi}\sum_{j=1}^{N}\int_{\mathbb{T}(0,\varrho_j)}|u(\zeta)|\mathrm{d}s(\zeta), \qquad z \in \mathbb{D}(0,\vartheta),$$

and

$$|\nabla u(z)| \le \frac{(N+2)\delta^{-N+1}}{(\vartheta-|z|)^2\pi}\sum_{j=1}^{N}\int_{\mathbb{T}(0,\varrho_j)}|u(\zeta)|\mathrm{d}s(\zeta), \qquad z \in \mathbb{D}(0,\vartheta).$$

4.3. **Hardy-Littlewood ellipticity.** The next lemma is an elaboration on a theme developed by Hardy and Littlewood in 1931, see Theorem 5 of [14] (see also [11], p. 121). They considered $N = 1$ (harmonic u), and observed first that for $1 \le p < +\infty$, the function $|u|^p$ is subharmonic, so that

$$|u(0)|^p \le \frac{1}{\pi}\int_{\mathbb{D}}|u|^p\mathrm{d}A.$$

For $0 < p < 1$, $|u|^p$ need not be subharmonic. However, Hardy and Littlewood found that the above inequality survives nevertheless, if the right hand side is multiplied by a suitable constant. This is an aspect of harmonic functions (and of the Laplacian) which we would like to call *Hardy-Littlewood ellipticity*. This fact was generalized to harmonic functions in $\mathbb{R}^n$, with $n > 2$, by Fefferman and Stein [7]. This Hardy-Littlewood ellipticity survives also in the context of polyharmonic functions.

Here is a 1994 result by Pavlović [31, Lemma 5], [30].

Lemma 4.2. $(0 < p < +\infty)$ *There exists a positive constant* $C_1(N,p)$ *depending only on N and p such that for all N-harmonic functions u on the unit disk* $\mathbb{D}$ *we have*

$$|u(z)|^p \le C_1(N,p)\int_{\mathbb{D}}|u|^p\mathrm{d}A, \qquad z \in \mathbb{D}(0,\tfrac{1}{2}).$$

Proof. Here we follow the elegant argument by Pavlović. One can develop an alternative argument based on a selection of optimal radii which shares some features with our previous work [4] (see, e.g., the proof of Theorem 8.4 in [21]).

In the case $1 \le p < +\infty$, the asserted estimate can be obtained in a rather straight-forward fashion based on the integral representation (4.7) (sketch: we let the points ϱ_j vary over short disjoint subintervals of $[\frac{3}{4}, 1[$, and integrate both sides with respect to all the ϱ_j over those short intervals. We then apply Hölder's inequality).

We now focus on the remaining case $0 < p < 1$. We quickly realize that it is enough to obtain the asserted estimate for a dilate u_r of u, where $u_r(z) = u(rz)$ and $0 < r < 1$. This allows us to suppose that u extends to be N-harmonic in a neighborhood of the closed unit disk. In particular, we may assume that u is bounded in $\mathbb{D}$.

We pick a $w \in \mathbb{D}$ such that

$$2|u(w)|^p(1-|w|^2)^2 \ge \max_{z\in\mathbb{D}} |u(z)|^p(1-|z|^2)^2.$$

Then, in view of the already established bound for $p = 1$ (see the first portion of this proof), we have the estimate (we write $\rho := \frac{1}{2}(1-|w|)$)

$$\begin{aligned}|u(w)|(1-|w|^2)^2 \le c_1(N,p)\int_{\mathbb{D}(w,\rho)}|u(z)|\mathrm{d}A(z) \le c_1(N,p)\max_{\mathbb{D}(w,\rho)}|u|^{1-p}\int_{\mathbb{D}(w,\rho)}|u)|^p\mathrm{d}A \\ \le c_2(N,p)|u(w)|^{1-p}\int_{\mathbb{D}}|u|^p\mathrm{d}A.\end{aligned}$$

It now follows that

$$|u(z)|^p \le c_3(N,p)|u(w)|^p(1-|w|^2)^2 \le c_4(N,p)\int_{\mathbb{D}}|u|^p\mathrm{d}A, \qquad z\in\mathbb{D}(0,\tfrac{1}{2}).$$

as claimed. Here, $c_j(N,p)$ stands for positive constants that only depend on N,p.

□

As a consequence, we obtain the following result of Pavlović:

Corollary 4.3. $(0 < p < +\infty)$ *There exists a positive constant $C_2(N,p)$ depending only on N and p such that for all N-harmonic functions u on the unit disk $\mathbb{D}$ we have*

$$|\nabla u(z)|^p \le C_2(N,p)\int_{\mathbb{D}}|u|^p\mathrm{d}A, \qquad z\in\mathbb{D}(0,\tfrac{1}{2}).$$

For future use, we formulate the above results in the setting of a general disk $\mathbb{D}(z_0, r)$.

Corollary 4.4. $(0 < p < +\infty)$ *Suppose u is N-harmonic in the disk $\mathbb{D}(z_0,r)$, where $z_0 \in \mathbb{C}$ and the radius r is positive. Then there exist positive constants $C_1(N,p)$ and $C_2(N,p)$ depending only on N and p such that*

$$|u(z)|^p \le \frac{C_1(N,p)}{r^2}\int_{\mathbb{D}(z_0,r)}|u|^p\mathrm{d}A, \qquad z\in\mathbb{D}(z_0,\tfrac{1}{2}r),$$

and

$$|\nabla u(z)|^p \le \frac{C_2(N,p)}{r^{2+p}}\int_{\mathbb{D}(z_0,r)}|u|^p\mathrm{d}A, \qquad z\in\mathbb{D}(z_0,\tfrac{1}{2}r).$$

Finally, Pavlović also obtains effective pointwise and integral bounds on u and ∇u given that $u \in \mathrm{PH}^p_{N,\alpha}(\mathbb{D})$.

Corollary 4.5. $(0 < p < +\infty)$ *Suppose u is N-harmonic in $\mathbb{D}$ and that α is a real parameter. Then there exist constants $C_3(N, p, \alpha)$ and $C_4(N, p, \alpha)$ which depend only on N, p, α, such that*

$$|u(z_0)|^p \le \frac{C_3(N, p, \alpha)}{(1 - |z_0|)^{\alpha+2}} \int_{\mathbb{D}} |u(z)|^p (1 - |z|^2)^\alpha \mathrm{d}A(z), \qquad z_0 \in \mathbb{D},$$

and

$$|\nabla u(z_0)|^p \le \frac{C_4(N, p, \alpha)}{(1 - |z_0|)^{\alpha+p+2}} \int_{\mathbb{D}} |u(z)|^p (1 - |z|^2)^\alpha \mathrm{d}A(z), \qquad z_0 \in \mathbb{D}.$$

Corollary 4.6. $(0 < p < +\infty)$ *Suppose u is N-harmonic in the unit disk $\mathbb{D}$ and that α is a real parameter. Then there exists a constant $C_5(N, p, \alpha)$ which depends only on N, p, α, such that*

$$\int_{\mathbb{D}} |\nabla u|^p (1 - |z|^2)^{\alpha+p} \mathrm{d}A(z) \le C_5(N, p, \alpha) \int_{\mathbb{D}} |u|^p (1 - |z|^2)^\alpha \mathrm{d}A(z).$$

Corollary 4.7. $(0 < p < +\infty)$ *If $u \in \mathrm{PH}^p_{N,\alpha}(\mathbb{D})$, then $\partial_z u$ and $\bar\partial_z u$ are both in $\mathrm{PH}^p_{N,\alpha+p}(\mathbb{D})$.*

Corollary 4.8. $(0 < p < +\infty)$ *If $u \in \mathrm{PH}^p_{N,\alpha}(\mathbb{D})$, then $\partial_z^j \bar\partial_z^k u \in \mathrm{PH}^p_{N,\alpha+(j+k)p}(\mathbb{D})$ for $j, k = 0, 1, 2, \ldots$.*

4.4. **Control of the antiderivative.** A particular instance of Corollary 4.7 is when f is holomorphic in $\mathbb{D}$: If $f \in L^p_\alpha(\mathbb{D})$ then $f' \in L^p_{\alpha+p}(\mathbb{D})$. In the converse direction, we have the following.

Proposition 4.9. $(0 < p < +\infty)$ *Suppose f is holomorphic in $\mathbb{D}$, with $f' \in L^p_{\alpha+p}(\mathbb{D})$ for some real parameter α. Then:*
(a) *If $\alpha \le -1 - p$, then f is constant.*
(b) *If $\alpha > -1$ and $1 \le p < +\infty$, then $f \in L^p_\alpha(\mathbb{D})$.*
(c) *If $-1 - p < \alpha \le -1$ and $1 \le p < +\infty$, then $f \in L^p_\beta(\mathbb{D})$ for every $\beta > -1$.*
(d) *If $\frac{1}{2} < p < 1$ and $-1 - p < \alpha \le p - 2$, then $f \in L^1_\beta(\mathbb{D}) \cap L^p_\beta(\mathbb{D})$ for every $\beta > -1$.*
(e) *If $0 < p < 1$ and $\alpha > p - 2$, then $f \in L^1_{-2+(\alpha+2)/p}(\mathbb{D})$. Moreover, the inclusion $L^1_{-2+(\alpha+2)/p}(\mathbb{D}) \subset L^p_{\alpha+1-p+\epsilon}(\mathbb{D})$ holds for every $\epsilon > 0$.*

Proof. If $\alpha \le -1 - p$ we must have $f'(z) \equiv 0$ which makes f constant. This settles part (a). As for parts (b) and (c), this follows from Proposition 1.11 of [21].

We turn to the case $0 < p < 1$. Holomorphic functions are harmonic, so Corollary 4.5 applied to $u = f'$ and $N = 1$ tells us that the function

$$(1 - |z|^2)^{\alpha+p+2} |f'(z)|^p$$

is uniformly bounded in the disk $\mathbb{D}$. It now follows from this and our assumption $f' \in L^p_{\alpha+p}(\mathbb{D})$ that

$$\int_{\mathbb{D}} |f(z)| (1 - |z|^2)^{\alpha+p+(\alpha+p+2)(1-p)/p} \mathrm{d}A(z) < +\infty,$$

which we simplify to $f' \in L^1_{-1+(\alpha+2)/p}(\mathbb{D})$. If $\alpha \le p - 2$, this gives that $f \in L^1_\beta(\mathbb{D})$ for every $\beta > -1$, by Proposition 1.11 of [21], which settles part (d), if we also use Hölder's inequality to obtain the L^p statement. If instead $\alpha > p - 2$, then Proposition 1.11 of [21] tells us that $f \in L^1_{-2+(\alpha+2)/p}(\mathbb{D})$, and part (e) follows, except for the inclusion. The inclusion is a simple consequence of Hölder's inequality. ∎

4.5. **Control of individual terms in the Almansi expansion.** With the help of Proposition 4.9, we may now obtain integral control of the individual terms of the Almansi expansion.

Corollary 4.10. $(0 < p < +\infty)$ *If* $u \in \mathrm{PH}^p_{N,\alpha}(\mathbb{D})$ *has the Almansi expansion*

$$u(z) = u_0(z) + |z|^2 u_1(z) + \cdots + |z|^{2N-2} u_{N-1}(z),$$

where the u_j *are harmonic in* $\mathbb{D}$*, then*
(a) *if* $1 \le p < +\infty$ *and* $\alpha > -1 - (N-1)p$*, then* $u_{N-1} \in L^p_{\alpha+(N-1)p}(\mathbb{D})$*, and*
(b) *if* $0 < p < 1$ *and* $\alpha \ge -1 - N + p$*, then* $u_{N-1} \in L^p_{\alpha+N-p+\epsilon}(\mathbb{D})$*, for every* $\epsilon > 0$.
(c) *if* $1 \le p < +\infty$ *and* $\alpha \le -1 - (N-1)p$*, or if* $0 < p < 1$ *and* $\alpha \le -1 - N + p$*, then* $u_{N-1} \in L^p_{-1+\epsilon}(\mathbb{D})$*, for every* $\epsilon > 0$.

Proof. We split $u_j = f_j + \overline{g_j}$, where f_j, g_j are holomorphic, with $g_j(0) = 0$. We calculate that

$$\partial_z \Delta^{N-1} u(z) = (N-1)! 4^{N-1} \partial_z^N [z^{N-1} f_{N-1}(z)],$$

and see from Corollary 4.8 that $\partial_z \Delta^{N-1} u$ is a holomorphic function in $L^p_{\alpha+(2N-1)p}(\mathbb{D})$. Integrating backwards using Proposition 4.9 shows that if $1 \le p < +\infty$ and $\alpha > -1 - (N-1)p$, then $f_{N-1} \in L^p_{\alpha+(N-1)p}(\mathbb{D})$, while if $0 < p < 1$ and $\alpha \ge -1 - N + p$, we instead have $f_{N-1} \in L^p_{\alpha+N-p+\epsilon}(\mathbb{D})$ for every $\epsilon > 0$. If instead $1 \le p < +\infty$ and $\alpha \le -1 - (N-1)p$ or $0 < p < 1$ and $\alpha \le -1 - N + p$, we find that $f_{N-1} \in L^p_{-1+\epsilon}(\mathbb{D})$ for every $\epsilon > 0$. Analogously, the function g_{N-1} has the same integrability properties, and then $u_{N-1} = f_{N-1} + \bar{g}_{N-1}$ automatically has the asserted properties. The proof is complete. □

4.6. **Divisibility of a polyharmonic functions by a canonical factor.** Some N-harmonic functions $u(z)$ are of the form $(1-|z|^2)\widetilde{u}(z)$, where $\widetilde{u}$ is $(N-1)$-harmonic. The following result offers an instance when this happens.

Proposition 4.11. $(0 < p < +\infty)$ *Suppose* $u \in \mathrm{PH}^p_{N,\alpha}(\mathbb{D})$ *for some integer* $N = 1, 2, 3, \ldots$*, and a real* α *with* $\alpha \le \min\{p-2, -1\}$*. Then if* $N \ge 2$*,* u *has the form* $u(z) = (1-|z|^2)\widetilde{u}(z)$*, where* $\widetilde{u} \in \mathrm{PH}^p_{N-1,\alpha+p}(\mathbb{D})$*, while if* $N = 1$*, we have that* $u(z) \equiv 0$.

Proof. We first consider the case $0 < p < 1$. Since then $\alpha \le p-2$, we have that $u \in \mathrm{PH}_{N,\alpha}(\mathbb{D}) \subset \mathrm{PH}_{N,p-2}(\mathbb{D})$, because

$$\|u\|^p_{p,p-2} = \int_{\mathbb{D}} |u(z)|^p (1-|z|^2)^{p-2} \mathrm{d}A(z) \le \int_{\mathbb{D}} |u(z)|^p (1-|z|^2)^{\alpha} \mathrm{d}A(z) = \|u\|^p_{p,\alpha} < +\infty, \tag{4.14}$$

and the pointwise estimate of Corollary 4.5 tells us that

$$[(1-|z|^2)|u(z)|]^p \le 2^p C_3(N, p, p-2)\, \|u\|^p_{p,p-2}, \qquad z \in \mathbb{D}. \tag{4.15}$$

We conclude from (4.14) and (4.15) that

$$\int_{\mathbb{D}} \frac{|u(z)|}{1-|z|^2} \mathrm{d}A(z) \le 2^{1-p} [C_3(N_0, p, p-2)]^{(1-p)/p} \|u\|_{p,\alpha} < +\infty, \tag{4.16}$$

which by an elementary argument involving polar coordinates implies that

$$\liminf_{\varrho \to 1^-} \int_{\mathbb{T}(0,\varrho)} |u| \mathrm{d}s = 0. \tag{4.17}$$

From the alternative Almansi representation (2.8), we have that

$$u(z) = v_0(z) + (1-|z|^2) v_1(z) + \cdots + (1-|z|^2)^{N-1} v_{N-1}(z),$$

where the functions v_j are all harmonic in $\mathbb{D}$. The extension of u can then be written as

$$\mathbf{E}[u](z,\varrho) = v_0(z) + (1-\varrho^2)v_1(z) + \cdots + (1-\varrho^2)^{N-1}v_{N-1}(z),$$

and by (4.3) combined with (4.17), we find that

$$|v_0(z)| = \lim_{\varrho\to 1^-} |\mathbf{E}[u](z,\varrho)| \le \liminf_{\varrho\to 1^-} \frac{1}{2\pi\varrho}\,\frac{\varrho+|z|}{\varrho-|z|}\int_{\mathbb{T}(0,\varrho)} |u(\zeta)|\mathrm{d}s(\zeta) = 0, \qquad z\in\mathbb{D}, \tag{4.18}$$

that is, $v_0(z)\equiv 0$. In case $N=1$, we are finished. In case $N\ge 2$, we note that by the alternative Almansi representation, this means that the function

$$\widetilde{u}(z) := \frac{u(z)}{1-|z|^2} = v_1(z) + (1-|z|^2)v_2(z)\cdots + (1-|z|^2)^{N-2}v_{N-1}(z),$$

is $(N-1)$-harmonic. It follows that $\widetilde{u}\in \mathrm{PH}^p_{N-1,\alpha+p}(\mathbb{D})$, as claimed.

We finally turn to the case $1\le p<+\infty$. Since $\alpha\le -1$, we have

$$\|u\|^p_{p,-1-p} = \int_{\mathbb{D}} \frac{|u(z)|^p}{1-|z|^2}\mathrm{d}A(z) \le \int_{\mathbb{D}} |u(z)|^p(1-|z|^2)^\alpha \mathrm{d}A(z) < +\infty.$$

An elementary argument now shows that

$$\liminf_{\varrho\to 1^-}\int_{\mathbb{T}(0,\varrho)} |u|^p\mathrm{d}s = 0$$

so that in particular (recall that $1\le p<+\infty$) – by Hölder's inequality –

$$\liminf_{\varrho\to 1^-}\int_{\mathbb{T}(0,\varrho)} |u|\mathrm{d}s = 0.$$

This is (4.17). We may then use (4.18) to conclude that $v_0(z)\equiv 0$, and if $N=1$, we are finished. If $N\ge 2$, we obtain instead that $u(z) = (1-|z|^2)\widetilde{u}(z)$ where $\widetilde{u}$ is $(N-1)$-harmonic. Then $\widetilde{u}\in\mathrm{PH}^p_{N-1,\alpha+p}(\mathbb{D})$, as claimed. □

4.7. **A criterion for the triviality of a polyharmonic function.** We obtain a sufficient criterion for $\mathrm{PH}^p_{N,\alpha}(\mathbb{D}) = \{0\}$.

Proposition 4.12. $(0<p<+\infty)$ *Suppose α is real with $\alpha\le -1-(2N-1)p$, for some integer $N=1,2,3,\ldots$. Then* $\mathrm{PH}^p_{N,\alpha}(\mathbb{D}) = \{0\}$.

Proof. As the spaces $\mathrm{PH}^p_{N,\alpha}(\mathbb{D})$ grow with α, it is enough to obtain the result when α is critically big: $\alpha = -1-(2N-1)p$.

Step 1. *We show that the assertion holds for $N=1$:* $\mathrm{PH}^p_{1,-1-p}(\mathbb{D}) = \{0\}$. To this end, we pick a function $v\in\mathrm{PH}^p_{1,-1-p}(\mathbb{D})$, and observe that since v is harmonic, $\partial_z v$ is holomorphic while $\bar\partial_z v$ is conjugate-holomorphic. Moreover, Corollary 4.7 tells us that $\partial_z v, \bar\partial_z v\in\mathrm{PH}^p_{1,-1}(\mathbb{D})$. Now $|\partial_z v|^p$ and $|\bar\partial_z v|^p$ are both subharmonic in $\mathbb{D}$, and in particular, their means on the circles $|z|=r$ increase with r, $0<r<1$. Using polar coordinates, then, we realize that

$$\|\partial_z v\|^p_{p,-1} + \|\bar\partial_z v\|^p_{p,-1} = \int_{\mathbb{D}} \left(|\partial_z v(z)|^p + |\bar\partial_z v(z)|^p\right)\frac{\mathrm{d}A(z)}{1-|z|^2} < +\infty$$

forces $\partial_z v = 0$ and $\bar\partial_z v = 0$, so that v must be constant. As the only constant function in $\mathrm{PH}^p_{1,-1-p}(\mathbb{D})$ is the zero function, we obtain $v=0$, and the conclusion $\mathrm{PH}^p_{1,-1-p}(\mathbb{D}) = \{0\}$ is immediate.

STEP 2. *We show that the assertion holds for $N>1$:* $\mathrm{PH}^p_{1,-1-(2N-1)p}(\mathbb{D})=\{0\}$. We pick a function $u\in\mathrm{PH}^p_{N,-1-(2N-1)p}(\mathbb{D})$ and intend to obtain that $u=0$. Since u is N-harmonic, the function $v_1:=\Delta^{N-1}u$ is harmonic. Moreover, as

$$v_1(z)=\Delta^{N-1}u(z)=4^{N-1}\partial_z^{N-1}\bar\partial_z^{N-1}u(z)$$

Corollary 4.8 now tells us that $v_1=\Delta^{N-1}u\in\mathrm{PH}^p_{1,-1-p}(\mathbb{D})$. This case we handled in Step 1, so that we know that $v_1=0$. But then the function $v_2:=\Delta^{N-2}u$ is harmonic, and by Corollary 4.8, $v_2=\Delta^{N-2}u\in\mathrm{PH}^p_{1,-1-3p}(\mathbb{D})=\{0\}$ because trivially $\mathrm{PH}^p_{1,-1-3p}(\mathbb{D})\subset\mathrm{PH}^p_{1,-1-p}(\mathbb{D})=\{0\}$. If $N=2$, we have arrived at the conclusion that $u=0$, as needed. If $N>2$, we continue, and form the successively the functions $v_j:=\Delta^{N-j}u$ which belong to $\mathrm{PH}^p_{1,-1-(2j-1)p}(\mathbb{D})=\{0\}$ for $j=3,\ldots,N$. With $j=N$ we arrive at $u=v_N=0$, as needed. □

5. THE WEIGHTED INTEGRABILITY STRUCTURE OF POLYHARMONIC FUNCTIONS

5.1. **A family of polyharmonic kernels.** For $N=1,2,3,\ldots$ and $j=0,1,\ldots,N$, let

$$U_{j,N}(z):=\frac{(1-|z|^2)^{N+j-1}}{|1-z|^{2j}}, \tag{5.1}$$

so that with

$$U_{j,j}(z)=\frac{(1-|z|^2)^{2j-1}}{|1-z|^{2j}},$$

we obtain the relation

$$U_{j,N}(z)=(1-|z|^2)^{N-j}U_{j,j}(z). \tag{5.2}$$

The functions $U_{j,N}$ are N-harmonic, and as we shall see, they are extremal for the critical integrability type $\beta(N,p)$. The function $U_{1,1}$ is the Poisson kernel for the boundary point at 1, and the function $U_{1,2}$ is known in the context of the bilaplacian as the *harmonic compensator* [18]. We also note that the function $U_{2,2}$ appears implicitly in the biharmonic setting in [1] ($U_{2,2}(z)=2F(z,1)-H(z,1)$ in their notation); cf. [28]. More recently, in [29] the kernels $U_{N,N}$ are shown to solve the Dirichlet problem in the disk $\mathbb{D}$ for a certain (singular) second order elliptic differential operator. It remains to substantiate that $U_{j,N}$ is N-harmonic.

Lemma 5.1. *For $N=1,2,3,\ldots$ and $j=0,1,\ldots,N$, the functions $U_{j,N}$ are all N-harmonic in $\mathbb{C}\setminus\{1\}$.*

Proof. By inspection, the function $U_{0,N}(z)=(1-|z|^2)^{N-1}$ is N-harmonic. As for the other kernels $U_{j,N}$ with $1\le j\le N$, the identity (5.2) together with the Almansi representation (2.2) shows that it is enough to know that $U_{j,j}$ is j-harmonic. Flipping variables, we just need to show that $U_{N,N}$ is N-harmonic. To this end, we change variables to $\zeta=1-z$ and see from the binomial theorem that

$$U_{N,N}(1-\zeta)=\frac{(\zeta+\bar\zeta-\zeta\bar\zeta)^{2N-1}}{\zeta^N\bar\zeta^N}=\sum_{j,k}(-1)^{j+k+1}\frac{(2N-1)!}{j!k!(2N-j-k-1)!}\zeta^{N-k-1}\bar\zeta^{N-j-1},$$

where j,k range over integers with $j,k\ge0$ and $j+k\le 2N-1$. It follows that $U_{N,N}$ is N-harmonic for $\zeta\neq0$, because in every term of the above sum, we have either $0\le N-k-1\le N-1$ or $0\le N-j-1\le N-1$, or both. An application of $\Delta^N_\zeta=4^N\partial^N_\zeta\bar\partial^N_\zeta$ to each term of the finite sum then then results in 0, as needed. □

We recall the definition of the functions $b_{j,N}(p)$ from (3.1) and (3.2).

Lemma 5.2. $(0 < p < +\infty)$ *For* $N = 1, 2, 3 \ldots$, $j = 0, \ldots, N$, *and real* α, *we have that*

$$U_{j,N} \in \mathrm{PH}^p_{N,\alpha}(\mathbb{D}) \iff \alpha > b_{j,N}(p).$$

Proof. To decide when $U_{j,N} \in \mathrm{PH}^p_{N,\alpha}(\mathbb{D})$, we calculate:

$$\|U_{j,N}\|^p_{p,\alpha} = \int_{\mathbb{D}} |U_{j,N}|^p (1-|z|^2)^\alpha \mathrm{d}A = \int_{\mathbb{D}} \frac{(1-|z|^2)^{(N+j-1)p+\alpha}}{|1-z|^{2jp}} \mathrm{d}A(z). \tag{5.3}$$

By Lemma 2.4, we have the following. For $j = 0$, $U_{j,N} = U_{0,N} \in \mathrm{PH}^p_{N,\alpha}(\mathbb{D})$ if and only if $(N-1)p+\alpha > -1$. For $1 \le j \le N$, however, $U_{j,N} \in \mathrm{PH}^p_{N,\alpha}(\mathbb{D})$ if and only if both $(N+j-1)p+\alpha > -1$ and $(N+j-1)p+\alpha > 2(jp-1)$. After some trivial algebraic manipulations, the assertion of the lemma is now immediate. □

So, if α meets

$$\alpha > \min_{j:0\le j\le N} b_{j,N}(p),$$

then one of the functions $U_{0,N}, U_{1,N}, \ldots, U_{N,N}$ will be in $\mathrm{PH}^p_{N,\alpha}(\mathbb{D})$, so that in particular,

$$\mathrm{PH}^p_{N,\alpha}(\mathbb{D}) \neq \{0\}.$$

It is quite remarkable that this criterion is also necessary for $\mathrm{PH}^p_{N,\alpha}(\mathbb{D}) \neq \{0\}$, as Theorem 3.1 says. The proof will be supplied in Section 7.

6. The structure of polyharmonic functions: the cellular decomposition

6.1. **The basic properties of the partial differential operators $\mathbf{L}_\theta$.** We recall that $\mathbf{L}_\theta$ is the partial differential operator given by (3.5),

$$\mathbf{L}_\theta[u](z) = (1-|z|^2)\Delta u(z) + 4\theta[z\partial_z u(z) + \bar{z}\bar{\partial}_z u(z)] - 4\theta^2 u(z),$$

and that $\mathbf{M}$ is the operator of multiplication by $1-|z|^2$. The basic operator identities satisfied by $\mathbf{L}_\theta$ are the following:

$$\Delta \mathbf{L}_\theta = \mathbf{L}_{\theta-1}\Delta, \tag{6.1}$$

and

$$\mathbf{L}_\theta \mathbf{M} = \mathbf{M}\mathbf{L}_{\theta-1} - 8\theta\mathbf{I}, \tag{6.2}$$

where $\mathbf{I}$ is the identity operator. To obtain (6.1), we calculate as follows:

$$\begin{aligned}\Delta\mathbf{L}_\theta[u](z) = \Delta\Big[(1-|z|^2)\Delta u(z) + 4\theta[z\partial_z u(z) + \bar{z}\bar{\partial}_z u(z)] - 4\theta^2 u(z)\Big]\\ = (1-|z|^2)\Delta^2 u(z) - 4z\partial_z\Delta u(z) - 4\bar{z}\bar{\partial}_z\Delta u(z) - 4\Delta u(z)\\ + 4\theta[2\Delta u(z) + z\partial_z\Delta u(z) + \bar{z}\bar{\partial}_z\Delta u(z)] - 4\theta^2\Delta u(z)\\ = (1-|z|^2)\Delta^2 u(z) + 4(\theta-1)[z\partial_z\Delta u(z) + \bar{z}\bar{\partial}_z\Delta u(z)] - 4(\theta-1)^2\Delta u(z) = \mathbf{L}_{\theta-1}\Delta u(z).\end{aligned}$$

To instead obtain (6.2), we calculate somewhat analogously:

$$\begin{aligned}\mathbf{L}_\theta\mathbf{M}[u](z) &= (1-|z|^2)\Delta[(1-|z|^2)u(z)] + 4\theta[z\partial_z((1-|z|^2)u(z)) + \bar z\bar\partial_z((1-|z|^2)u(z))]\\ &- 4\theta^2(1-|z|^2)u(z) = (1-|z|^2)[(1-|z|^2)\Delta u(z) - 4z\partial_z u(z) - 4\bar z\bar\partial_z u(z) - 4u(z)]\\ &+ 4\theta[-2|z|^2u(z) + (1-|z|^2)(z\partial_z u(z) + \bar z\bar\partial_z u(z))] - 4\theta^2(1-|z|^2)u(z)\\ &= (1-|z|^2)^2\Delta u(z) + 4(\theta-1)(1-|z|^2)[z\partial_z u(z) + \bar z\bar\partial_z u(z)] - 4(\theta-1)^2(1-|z|^2)u(z)\\ &- 8\theta u(z) = (1-|z|^2)\mathbf{L}_{\theta-1}[u](z) - 8\theta u(z).\end{aligned}$$

By iteration of the operator identity (6.1), we obtain more generally that

$$\Delta^j\mathbf{L}_\theta = \mathbf{L}_{\theta-j}\Delta^j, \qquad j = 0,1,2,\ldots. \tag{6.3}$$

If we instead iterate (6.2), we find the following operator identity:

$$\mathbf{L}_\theta\mathbf{M}^j = \mathbf{M}^j\mathbf{L}_{\theta-j} + 4j(j-1-2\theta)\mathbf{M}^{j-1}, \qquad j = 0,1,2,\ldots. \tag{6.4}$$

Proposition 6.1. *We have the following factorization:*

$$\mathbf{L}_0\mathbf{L}_1\cdots\mathbf{L}_{n-1} = \mathbf{M}^n\Delta^n, \qquad n = 1,2,3,\ldots.$$

Proof. We argue by induction. Since $\mathbf{L}_0 = \mathbf{M}\Delta$, the assertion holds trivially for $n = 1$. Next, suppose we have established the identity for $n = n_0 \ge 1$, that is,

$$\mathbf{L}_0\mathbf{L}_1\cdots\mathbf{L}_{n_0-1} = \mathbf{M}^{n_0}\Delta^{n_0}$$

holds. We then see that

$$\mathbf{L}_0\mathbf{L}_1\cdots\mathbf{L}_{n_0} = \mathbf{M}^{n_0}\Delta^{n_0}\mathbf{L}_{n_0} = \mathbf{M}^{n_0}\mathbf{L}_0\Delta^{n_0} = \mathbf{M}^{n_0+1}\Delta^{n_0+1},$$

where we used the iterated operator identity (6.3) and that $\mathbf{L}_0 = \mathbf{M}\Delta$. The proof is complete. □

Corollary 6.2. *Fix a positive integer n. If v solves $\mathbf{L}_{n-1}[v] = 0$ in $\mathbb{D}$, then v is n-harmonic in $\mathbb{D}$.*

Corollary 6.3. *Fix a positive integer n. If v is n-harmonic in $\mathbb{D}$, then $\mathbf{L}_{n-1}[u]$ is $(n-1)$-harmonic. If $n = 1$, this should be interpreted as $\mathbf{L}_0[u] = 0$.*

6.2. **Mapping properties of $\mathbf{L}_\theta$.** We need to understand what space $\mathbf{L}_\theta$ maps $\mathrm{PH}^p_{N,\alpha}(\mathbb{D})$ into.

Proposition 6.4. $(0 < p < +\infty)$ *Suppose $u \in \mathrm{PH}^p_{N,\alpha}(\mathbb{D})$, for some integer $N = 1,2,3,\ldots$ and a real parameter α. Then, for θ real, we have $\mathbf{L}_\theta[u] \in \mathrm{PH}^p_{N,\alpha+p}(\mathbb{D})$.*

Proof. In view of Corollary 4.8, we have $\partial_z u, \bar\partial_z u \in \mathrm{PH}^p_{N,\alpha+p}(\mathbb{D})$ and $\Delta u = 4\partial_z\bar\partial_z u \in \mathrm{PH}^p_{N-1,\alpha+2p}(\mathbb{D})$. Here, we used that Δu is $(N-1)$-harmonic, which is immediate because u is N-harmonic. We remark that as a consequence, the function $\mathbf{M}\Delta u(z) = (1-|z|^2)\Delta u(z)$ is N-harmonic, and then $\mathbf{M}\Delta u \in \mathrm{PH}^p_{N,\alpha+p}(\mathbb{D})$. The assertion now follows. □

6.3. **The work of Olofsson.** Given a function v on $\mathbb{D}$, For $0 < r < 1$, we let v_r denote its *dilate*, the function on $\mathbb{T}$ given by $v_r(\zeta) := v(r\zeta)$. Recently, Olofsson obtained the following result [29].

Proposition 6.5. $(0 < p < +\infty,\ -\frac{1}{2} < \theta < +\infty)$ *For a continuous function $v : \mathbb{D} \to \mathbb{C}$, the following are equivalent:*
(i) *The function v solves $\mathbf{L}_\theta[v] = 0$ on $\mathbb{D}$, and its dilates v_r converge to a distribution as $r \to 1^-$, and*
(ii) *The function v is of the form*

$$v(z) = \frac{1}{2\pi}\int_{\mathbb{T}} \frac{(1-|z|^2)^{2\theta+1}}{|1-z\bar\xi|^{2\theta+2}} f(\xi)\,\mathrm{d}s(\xi), \qquad z \in \mathbb{D},$$

for some distribution f on $\mathbb{T}$, where the integral is understood in the sense of distribution theory.

We now indicate how to derive Proposition 3.8 from the above Proposition 6.5.

Proof of Proposition 3.8. We realize that Proposition 3.8 follows from Proposition 6.5 once we know that adding the following requirements in the setting of Proposition 6.5(i) forces the function v to have convergent dilates v_r as $r \to 1^-$ in the sense of distribution theory: (1) $\theta = n-1$ is a nonnegative integer, and (2) $v \in L^p_\alpha(\mathbb{D})$. First, we observe that since $\mathbf{L}_{n-1}[v] = \mathbf{L}_\theta[v] = 0$ is assumed to hold, we know from Corollary 6.2 that v is n-harmonic, so that $v \in \mathrm{PH}^p_{n,\alpha}(\mathbb{D})$. Next, by suitably iterating Corollary 4.10 and by combining the result with the pointwise bound of Corollary 4.5, we realize that all the harmonic functions u_j in the Almansi representation of $v \in \mathrm{PH}^p_{n,\alpha}(\mathbb{D})$,

$$u(z) = u_0(z) + |z|^2 u_1(z) + \cdots + |z|^{2n-2} u_{n-1}(z),$$

have the growth bound

$$|u_j(z)| = \mathrm{O}\Big((1-|z|)^{-\Lambda}\Big) \quad \text{as } |z| \to 1^-,$$

for some big positive constant Λ (which may depend on n, p, α). By a standard result in distribution theory, this means that each u_j is the Poisson integral of a distribution f_j on $\mathbb{T}$, and that the dilates $u_{j,r}(\zeta) := u_j(r\zeta)$ (for $\zeta \in \mathbb{T}$) converge to a nonzero constant multiple of f_j in the sense of distribution theory as $r \to 1^-$. Then the dilates of u_r of u converge in the sense of distribution theory, as needed. □

6.4. **The proof of the cellular decomposition theorem.** We turn to the proof of Theorem 3.4.

Proof of Theorem 3.4. Uniqueness. We first treat the uniqueness part. We have the equation

$$w_0 + \mathbf{M}[w_1] + \cdots + \mathbf{M}^{N-1}[w_{N-1}] = 0, \tag{6.5}$$

where the functions w_j are $(N-j)$ harmonic and solve the partial differential equation $\mathbf{L}_{N-j-1}[w_j] = 0$ on $\mathbb{D}$. We need to show that all the functions w_j vanish, where $j = 0, \ldots, N-1$. We resort to an induction argument. Clearly, when $N = 1$, the equation (6.5) just says $w_0 = 0$, as needed. Next, in the induction step we suppose that uniqueness holds for $N = N_0$, and intend to demonstrate that it must then also hold for $N = N_0 + 1$. The equation (6.5) with $N = N_0 + 1$ reads

$$\sum_{j=0}^{N_0} \mathbf{M}^j[w_j] = 0,$$

where the functions w_j solve $\mathbf{L}_{N_0-j}[w_j] = 0$. We now apply the operator $\mathbf{L}_{N_0}$ to both sides, and rewrite the equation using (6.4):

$$\sum_{j=0}^{N_0} \Big\{\mathbf{M}^j \mathbf{L}_{N_0-j}[w_j] + 4j(j-2N_0-1)\mathbf{M}^{j-1}[w_j]\Big\} = 0.$$

If we use the given information that the functions w_j solve $\mathbf{L}_{N_0-j}[w_j] = 0$, the above equation simplifies pleasantly:

$$\sum_{j=0}^{N_0-1} (j+1)(j-2N_0)\mathbf{M}^j[w_{j+1}] = 0.$$

By introducing the functions $\tilde{w}_j := (j+1)(j-2N_0)w_{j+1}$, the equation simplifies further:

$$\sum_{j=0}^{N_0-1} \mathbf{M}^j[\tilde{w}_j] = 0.$$

We observe that $\mathbf{L}_{N_0-j-1}[\tilde{w}_j] = 0$, and we are in the setting of $N = N_0$, and by the induction hypothesis, we have that $\tilde{w}_j = 0$ for all $j = 0, \ldots, N_0 - 1$. As a consequence, $w_j = 0$ for all $j = 1, \ldots, N_0$, and $w_0 = 0$ is then immediate from (6.5). This completes the uniqueness part of the proof.

Existence. Next, we turn to the existence part. We argue by induction in N. We first observe that the assertion is trivial for $N = 1$. In the induction step, we assume that the assertion of the theorem holds for $N = N_0 \geq 1$, and attempt to show that it must then also hold for $N = N_0 + 1$. The function u is now $(N_0 + 1)$-harmonic, and and we form the associated function $\mathbf{L}_{N_0}[u]$, which is then N_0-harmonic, by Corollary 6.3. Since $u \in \mathrm{PH}^p_{N_0+1,\alpha}(\mathbb{D})$, Proposition 6.4 gives that $\mathbf{L}_{N_0}[u] \in \mathrm{PH}^p_{N_0,\alpha+p}(\mathbb{D})$. By the induction hypothesis, then, we know that

$$\mathbf{L}_{N_0}[u] = \sum_{j=0}^{N_0-1} \mathbf{M}^j[h_j],$$

where h_j are $(N_0 - j)$-harmonic and solve $\mathbf{L}_{N_0-j-1}[h_j] = 0$. Moreover, again by the induction hypothesis, we have $\mathbf{M}^j[h_j] \in \mathrm{PH}^p_{N_0,\alpha+p}(\mathbb{D})$, so that by (3.7), $h_j \in \mathrm{PH}^p_{N_0-j,\alpha+(1+j)p}(\mathbb{D})$. We form the associated function

$$H := \frac{1}{4}\sum_{j=0}^{N_0-1} \frac{1}{(j+1)(2N_0-j)}\mathbf{M}^{j+1}[h_j],$$

and observe that $H \in \mathrm{PH}^p_{N_0+1,\alpha}(\mathbb{D})$. So the sum $u + H$ is in $\mathrm{PH}^p_{N_0+1,\alpha}(\mathbb{D})$, and we calculate that

$$\mathbf{L}_{N_0}[u+H] = \mathbf{L}_{N_0}[u] + \mathbf{L}_{N_0}[H] = \sum_{j=0}^{N_0-1}\left\{\mathbf{M}^j[h_j] + \frac{1}{4(j+1)(2N_0-j)}\mathbf{L}_{N_0}\mathbf{M}^{j+1}[h_j]\right\}$$
$$= \sum_{j=0}^{N_0-1}\left\{\mathbf{M}^j[h_j] + \frac{1}{4(j+1)(2N_0-j)}\Big(\mathbf{M}^{j+1}\mathbf{L}_{N_0-j-1}[h_j] - 4(j+1)(2N_0-j)\mathbf{M}^j[h_j]\Big)\right\} = 0,$$

where we used the operator identity (6.4) and that $\mathbf{L}_{N_0-j-1}[h_j] = 0$. So, with $w_0 := u + H$ and

$$w_j := -\frac{1}{4j(2N_0-j+1)}h_{j-1}, \qquad j = 1, \ldots, N_0,$$

we see that

$$u = \sum_{j=0}^{N_0} \mathbf{M}^j[w_j],$$

where w_j is (N_0+1-j)-harmonic with $\mathbf{L}_{N_0-j}[w_j] = 0$, for $j = 0, \ldots, N_0$. Moreover, the given integrability properties of the functions h_j lead to $w_j \in \mathrm{PH}^p_{N_0+1-j,\alpha+jp}(\mathbb{D})$ and $\mathbf{M}^j[w_j] \in \mathrm{PH}^p_{N_0+1,\alpha}(\mathbb{D})$. The existence of the asserted expansion has now been obtained. The proof is complete. □

7. Applications of the cellular decomposition

7.1. **Triviality of individual terms in the cellular decomposition.** We now analyze each term in the cellular decomposition separately. We recall the definition of the functions $a_{j,N}(p)$ from Section 3.

Proposition 7.1. $(0 < p < +\infty)$ *Fix integers $N = 1, 2, 3, \ldots$ and $j = 0, \ldots, N-1$. Suppose $u \in \mathrm{PH}^p_{N,\alpha}(\mathbb{D})$ is of the form $u = \mathbf{M}^j[w]$, where w is $(N-j)$-harmonic and $\mathbf{L}_{N-j-1}[w] = 0$. Then if $\alpha \leq a_{N-j,N}(p)$, we have that $u = 0$.*

Proof. By (3.7), the function w is in $\mathrm{PH}^p_{N-j,\alpha+jp}(\mathbb{D})$. From (3.9), we know that

$$\alpha + jp \le a_{N-j,N}(p) + jp = a_{N-j,N-j}(p),$$

so if we introduce $N' := N - j$ and $\alpha' := \alpha + jp$, it will be sufficient to show the following: *If $w \in \mathrm{PH}^p_{N',\alpha'}(\mathbb{D})$ solves $\mathbf{L}_{N'-1}[w] = 0$, and $\alpha' \le a_{N',N'}(p)$, then $w = 0$.* We recall from (3.1) and (3.8) that

$$a_{N',N'}(p) = \min\{b_{N',N'}(p), -1\}, \qquad b_{N',N'}(p) = \min\{-1 - (2N'-1)p, -2 + p\},$$

so that

$$a_{N',N'}(p) = b_{N',N'}(p) = -1 - (2N'-1)p \quad \text{for} \quad 0 < p \le \frac{1}{2N'}.$$

The assertion $w = 0$ is immediate from Proposition 4.12 in case $0 < p \le 1/(2N')$. Next, we consider the remaining interval $1/(2N') < p < +\infty$. Since

$$a_{N',N'}(p) = \min\{p - 2, -1\} \quad \text{for} \quad \frac{1}{2N'} < p < +\infty,$$

the assumption $\alpha \le a_{N',N'}(p)$ entails in view of Proposition 4.11 that $w = \mathbf{M}[\tilde{w}]$, where $\tilde{w}$ is $(N'-1)$-harmonic, which means that $\tilde{w} = 0$ if $N' = 1$. In view of Theorem 3.4, the function $\tilde{w} \in \mathrm{PH}^p_{N'-1,\alpha+p}(\mathbb{D})$ then has a unique expansion

$$\tilde{w} = \sum_{j=0}^{N'-2} \mathbf{M}^j[g_j],$$

where $\mathbf{M}^j[g_j] \in \mathrm{PH}^p_{N'-1,\alpha+p}(\mathbb{D})$ and g_j is $(N-j-1)$-harmonic with $\mathbf{L}_{N'-j-2}[g_j] = 0$. This means that $w = \mathbf{M}[\tilde{w}]$ has the expansion

$$w = \mathbf{M}[\tilde{w}] = \sum_{j=1}^{N'-1} \mathbf{M}^j[g_{j-1}] = \sum_{j=1}^{N'-1} \mathbf{M}^j[\tilde{g}_j], \tag{7.1}$$

where the terms are all in $\mathrm{PH}^p_{N',\alpha}(\mathbb{D})$, and $\mathbf{L}_{N'-j-1}[\tilde{g}_j] = 0$; here we have introduced $\tilde{g}_j := g_{j-1}$. From the uniqueness of the cellular decomposition in Theorem 3.4, we realize that (7.1) is only possible if $w = 0$. The proof is complete. □

Proposition 7.2. $(0 < p < +\infty)$ *Fix integers $N = 1, 2, 3, \ldots$ and $j = 0, \ldots, N-1$. Suppose α is real with $\alpha > a_{N-j,N}(p)$. Then there exists a nontrivial $u \in \mathrm{PH}^p_{N,\alpha}(\mathbb{D})$ of the form $u = \mathbf{M}^j[w]$, where w is $(N-j)$-harmonic with $\mathbf{L}_{N-j-1}[w] = 0$.*

Proof. For $0 < p \le 1$, we can use the function $u = U_{N-j,N} = \mathbf{M}^j[U_{N-j,N-j}]$, as given by (5.1), so that $w = U_{N-j,N-j}$. By Proposition 6.5, the function $w = U_{N-j,N-j}$ solves $\mathbf{L}_{N-j-1}[w] = 0$, by Lemma 5.2, u is in $\mathrm{PH}^p_{N,\alpha}(\mathbb{D})$ if and only if $\alpha > b_{N-j,N}(p)$, and we have $a_{N-j,N}(p) = b_{N-j,N}(p)$ for $0 < p \le 1$. For $1 \le p < +\infty$, we need to consider instead the function $u = \mathbf{M}^j[w]$, where

$$w(z) = (1 - |z|^2)^{2N-2j-1} \int_{\mathbb{T}} |1 - z\bar{\xi}|^{-2N+2j} \frac{\mathrm{d}s(\xi)}{2\pi}, \qquad z \in \mathbb{D}, \tag{7.2}$$

which solves $\mathbf{L}_{N-j-1}[w] = 0$, by a second application of Proposition 6.5. The function w given by (7.2) is bounded in the disk $\mathbb{D}$, so that $u = \mathbf{M}^j[w]$ is in $\mathrm{PH}^p_{N,\alpha}(\mathbb{D})$ for

$$\alpha > -1 - pj = a_{N-j,N}(p), \qquad 1 \le p < +\infty.$$

The proof is complete. □

7.2. **Characterization of the admissible region.** We are now ready to supply the proof of Theorem 3.1.

Proof of Theorem 3.1. We first observe that

$$\min_{j:1\le j\le N} a_{j,N}(p) = \min_{j:0\le j\le N} b_{j,N}(p).$$

After that, we understand that the assertion is a consequence of the cellular decomposition of Theorem 3.4 combined with Propositions 7.1 and 7.2. □

7.3. **The entangled region.** It remains to supply the proof of Proposition 3.6.

Proof of Proposition 3.6. In terms of the cellular decomposition of Theorem 3.4, it is a matter of deciding for which (p, α) the function w_{N-1} must equal 0. This is easy to do using Propositions 7.1 and 7.2. □

7.4. **The principal unentangled cell.** It remains to supply the proof of Proposition 3.7.

Proof of Proposition 3.7. In terms of the cellular decomposition of Theorem 3.4, it is a matter of deciding for which (p, α) the functions w_j, with $j = 0, \ldots, N-2$, must all equal 0. This is easy to do using Propositions 7.1 and 7.2. □

7.5. **The cellular decomposition for the general admissible cell.** It remains to supply the proof of Theorem 3.10.

Proof of Theorem 3.10. The criteria of the theorem check which terms actually occur in the cellular decomposition of Theorem 3.4, in accordance with Propositions 7.1 and 7.2. □

8. Concluding remarks

8.1. **Boundary effects.** It would be interesting to find out necessary and sufficient conditions on a distribution f on $\mathbb{T}$ in order to have that the potential

$$\mathbf{U}_{N,N}[f](z) = (1-|z|^2)^{2N-1}\int_{\mathbb{T}} |1 - z\bar{\xi}|^{-2N}\frac{\mathrm{d}s(\xi)}{2\pi},$$

belongs to, say, $\mathrm{PH}^p_{N,\alpha}(\mathbb{D})$ for $0 < p < 1/(2N)$ and $\alpha > b_{N,N}(p)$ close to $b_{N,N}(p)$. It is obvious that a sum of point masses at points of $\mathbb{T}$ with coefficients from ℓ^p gives rise to such a distribution f. However, it is easy to see that the functions $U_{N,N}(\gamma z)$ depend continuously on $\gamma \in \mathbb{T}$ in the space $\mathrm{PH}^p_{N,\alpha}(\mathbb{D})$, so the correct answer is not the trivial one (sums of point masses at points of $\mathbb{T}$ with coefficients from ℓ^p).

8.2. **Sharpening of some results.** It is possible to sharpen the assertion of Theorem 3.1 to arrive at the following results (cf. Section 4 of [2]).
(i) Suppose $0 < p < +\infty$, and that u is N-harmonic in $\mathbb{D}$. If

$$\liminf_{r\to 1^-}\ (1-r)^{\beta(N,p)}\int_{\mathbb{D}\setminus\mathbb{D}(0,r)} |u|^p\,\mathrm{d}A = 0,$$

then $u(z) \equiv 0$.
(ii) If $0 < p < \frac{1}{2N-1}$, and if

$$\liminf_{r\to 1^-}\frac{\int_{\mathbb{D}\setminus\mathbb{D}(0,r)} |u|^p \mathrm{d}A}{\int_{\mathbb{D}\setminus\mathbb{D}(0,r)} |U_{N,N}|^p\,\mathrm{d}A} = 0,$$

then $u(z) \equiv 0$.

Here, it is useful to know that

$$\int_{\mathbb{D}\setminus\mathbb{D}(0,r)} |U_{N,N}(z)|^p \mathrm{d}A(z) \asymp \begin{cases} (1-r)^{2-p}, & \text{if } p > \frac{1}{2N}, \\ (1-r)^{2-p}\log\frac{1}{1-r}, & \text{if } p = \frac{1}{2N}, \\ (1-r)^{1+(2n-1)p}, & \text{if } 0 < p < \frac{1}{2N}. \end{cases}$$

8.3. **An example of a cellular decomposition.** Let u be a real-valued biharmonic function. The Almansi representation permits us to find two holomorphic functions f_1, f_2 in the disk $\mathbb{D}$ with $\operatorname{Im} f_1(0) = \operatorname{Im} f_2(0) = 0$, such that

$$u(z) = \operatorname{Re} f_1(z) + |z|^2 \operatorname{Re} f_2(z), \qquad z \in \mathbb{D}.$$

Suppose that $0 < p < \frac{1}{3}$, $-1-3p < \alpha \leq -1-2p$, and that $u \in \mathrm{PH}^p_{2,\alpha}(\mathbb{D})$, so that we are in the entangled region (see Figure 3.1). Theorem 3.10 can be interpreted as saying that f_1, f_2 are related by the formula

$$z(f_1'(z) + f_2'(z)) - f_1(z) + f_2(z) = 0.$$

8.4. **Polyanalytic functions.** In the polyanalytic case the corresponding critical exponent is $\beta = -1-(N-1)p$, which may be interpreted as saying that no entanglement takes place. To be more precise, let $\mathrm{PA}^p_{N,\alpha}(\mathbb{D})$ denote the subspace of $L^p_\alpha(\mathbb{D})$ consisting of N-analytic functions, which by definition solve the partial differential equation $\bar\partial_z^N f = 0$ on $\mathbb{D}$. Then

$$\mathrm{PA}^p_{N,\alpha}(\mathbb{D}) = \{0\} \iff \alpha \leq -1-(N-1)p.$$

We explain the necessary argument for $N = 2$. Then $f \in \mathrm{PA}^p_{2,\alpha}(\mathbb{D})$ decomposes into $f(z) = f_1(z) + \bar z f_2(z)$ with holomorphic f_1, f_2, so that

$$zf(z) = zf_1(z) + |z|^2 f_2(z)$$

is biharmonic. We form the extension of zf, $\mathbf{E}[zf](z,\varrho) = zf_1(z) + \varrho^2 f_2(z)$, so that $\mathbf{E}[zf](z,\varrho) = zf(z)$ for $z \in \mathbb{T}(0,\varrho)$. Now, if

$$\int_{\mathbb{D}} |zf(z)|^p (1-|z|^2)^{-1-p} \mathrm{d}A(z) < +\infty,$$

an elementary argument shows that

$$\liminf_{\varrho\to 1^-} (1-\varrho^2)^{-p} \int_{\mathbb{T}(0,\varrho)} |\mathbf{E}[zf](\zeta,\varrho)|^p \mathrm{d}s(\zeta) = \liminf_{\varrho\to 1^-} (1-\varrho^2)^{-p} \int_{\mathbb{T}(0,\varrho)} |\zeta f(\zeta)|^p \mathrm{d}s(\zeta) = 0.$$

As the function $|\mathbf{E}[zf](\cdot,\varrho)|^p$ is subharmonic, we have that

$$|\mathbf{E}[zf](z,\varrho)|^p \leq \frac{1}{2\pi\varrho} \frac{\varrho+|z|}{\varrho-|z|} \int_{\mathbb{T}(0,\varrho)} |F_\varrho(\zeta)|^p \mathrm{d}s(\zeta), \qquad z \in \mathbb{D}(0,\varrho),$$

and a combination of the above tells us that $\mathbf{E}[zf](z,1) = zf_1(z) + f_2(z) \equiv 0$. We arrive at $zf(z) = z(1-|z|^2)f_1(z)$, and the problem reduces to $N = 1$, $\alpha \leq -1$, which is trivial.

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